III Parabolic case.

Lemma: let $f: X \circlearrowleft$ with a parabolic (tangent to the identity) fixed point p. Denote by $\Phi: P \to \mathbb{C}$ denote the Fatou component of an attracting petal P, attached to the attracting direction v.

Then Φ extends uniquely to the attracting basin $\mathcal{A}_{p,v}$ satisfying the Abel equation $\Phi(f(z)) = \Phi(z) + 1$.

Proof: It suffices to set $\Phi(z) = \Phi(f^n(z)) - n$ for n big enough.

This formula has to be satisfied, which implies the unicity, while Φ is well defined since it does not depend on the choice of n big enough. $\square$

Lemma
Corollary: In the repelling case: P a repelling petal, then $\Phi^{-1}: \Phi(P) \to P$ extends uniquely to a globally defined holomorphic map $\Psi: \mathbb{C} \to X$ satisfying $f(\Psi(w)) = \Psi(w+1)$.

Proof: similar to before. $\psi(z) = f^n(\psi(w-n))$ for $n >> 0$. $\square$

Theorem: let $f: \hat{\mathbb{C}} \circlearrowleft$, $\deg f \geq 2$, and $p \in \hat{\mathbb{C}}$ a fixed point with $f'(p) = 1$.

Then each immediate basin of attraction for p contains at least a critical point. Moreover, in any such basin there exist a maximal petal of the form $\Phi^{-1}(H_R) =: P$

(Maximal = minimal R, $H_R = \{z \mid \mathrm{Re}\, z > R\}$.) so that $\Phi: P \to H_R$ is a biholomorphism. Then ∂P contains at least a critical point of f.

Proof: The proof is completely analogous to the attracting case:

1) try to extend $\Phi^{-1} = \psi$ on horizontal halflines $\{t + iy \mid t > t_0\}$.

One cannot do it indefinitely, or we would have that $\Psi(\mathbb{C})$ is

a simply connected parabolic surface in $\hat{\mathbb{C}}$, which is hence $\hat{\mathbb{C}} \setminus \{q\}$.
This implies $d=1$, contradiction.

2) Taking R ~~maximal~~ so that Ψ is defined on H_R, we show that $\psi(\partial H_R)$ has at least a critical point.

If not, we could extend ψ to $H_{R'}$, $R'<R$, by setting $\psi(w) = g(\psi(w+1))$ where g is a local inverse of f at $f(z_0)$, $z_0 \in \psi(\partial H_R)$.

Notice that for $|\mathrm{Im}(\phi(z_0))| \geq S > 0$, we are in the area where the extension done in the local setting holds, and Ψ extends automatically there.

By compactness of $\{R + yi, |y| < S\}$, we get the statement. $\square$

One can as well show that $\phi : \overline{P} \to \overline{H_R}$ is a homeomorphism.

Similar in the case of a periodic parabolic point, we can find a $q \in \mathbb{N}$ so that z_0 is a fixed point for f^q, and $(f^q)'(z_0) = 1$.

The multiplicity is $r+1$ with $r = qs$, $s \in \mathbb{N}$, q ~~minimal~~ with the above property. It follows that the basins of attractions of the cycle z_0 (all cumulatively) have s critic points for f.

We deduce:

Corollary. Let $f : \hat{\mathbb{C}} \circlearrowleft$ a rational map of degree $d \geq 2$. Then the number of contracting cycles + the number of parabolic cycles, is $\leq 2d-2$.

Proof. $2d-2 = \# \mathcal{C}(f)$, and for each cycle, there is at least one critical point on the basin of attraction. Finally, attraction basins are disjoint. $\square$

Rem: parabolic points and superattracting points are counted with multiplicity.
Examples realising the bound: z^d; $z + z^d$.

IV Irrational case

Recall that the post-critical set $PC(f)$ is defined as $PC(f) = \bigcup_{n\geq 1} f^n(C(f))$

Theorem: $f: \hat{\mathbb{C}} \circlearrowleft$ rational map of degree $d \geq 2$.

- Every Cremer periodic cycle is contained in $\overline{PC(f)}$.
- The boundary of every (cycle of) Siegel disk(s) is contained in $\overline{PC(f)}$.

Proof: Set $U = \hat{\mathbb{C}} \setminus \overline{PC(f)}$, and $V = f^{-1}(U)$. ($P = PC(f)$).

Since $f^{-1}(P) \supset P \leadsto f^{-1}(\overline{P}) \supset \overline{P} \Rightarrow V \subset U$.

Since U has no critical values, hence V has no critical points,

$f: V \to U$ is a $d:1$ covering map

(More precisely, for any $V' \subset V$ connected component, $f|_{V'}: V' \to f(V')$ is a covering map ($f(V')$ connected component of U).

We may assume that $\# \overline{P} \geq 3$.

If not: U is a parabolic Riemann surface, and being $f: V \xrightarrow{d:1} U$ a covering with $V \subset U$, we get $V = U$, which implies $f^{-1}(P) = P$. ($\Rightarrow P \subseteq \mathcal{E}$)

Up to change of coordinates, either $P = \{\infty\}$, or $P = \{\infty, 0\}$.

In the first case, f is a polynomial with no critical points in $\mathbb{C}$ (contradiction)

In the second case, f^2 is a polynomial with only a fixed critical point at $0 \leadsto f^2(z) = z^{d^2}$ ($\Rightarrow f(z) = z^{\pm d}$).

In this case, there are no periodic points with multiplier of the form $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and there is nothing to prove.

(preperiodic for $z^d \Leftrightarrow z = 0$, $z = \infty$, or $z = e^{2\pi i \frac{p}{q}}$, and $f'(z) = d z^{d-1}$ repelling.)

We may consider any connected component of V or U to be hyperbolic

Consider now a fixed point $z_0 = f(z_0) \in U$ $(\Rightarrow z_0 \in V = f^{-1}(U))$.

let $V_0 \subset U_0$ be the connected components containing z_0. $U_0 \supset f(V_0)$

~~#~~ We have two cases

1) $V_0 = U_0$. Then $f(V_0) \subset V_0$ and $V_0 \subset \text{Fatou}(f)$

(if it intersects $J(f)$ $\bigcup f^n(V_0) \supset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$). In this case z_0 is not a Cremer point.

2) $V_0 \subsetneq U_0$. Then $i: V_0 \hookrightarrow U_0$ strictly decreases Poincaré distances:

$$\rho_{V_0}(x,y) > \rho_{U_0}(x,y). \quad \forall x \neq y,\ x,y \in V_0.$$

Being $f: V \to U$ a covering map, it follows that also $f: V_0 \to U_0$ is a covering map (it suffices to show that $f(V_0) = U_0$, and to do so, you can check that $f(V_0)$ is both open and closed in U_0), so that

$$\rho_{U_0}(f(x), f(y)) \underset{\substack{\uparrow \\ \text{for } x,y \text{ close enough}}}{=} \rho_{V_0}(x,y) > \rho_{U_0}(x,y).$$

It follows that z_0 must be a repelling fixed point, hence not Cremer.

This shows the first property. We now focus with a Siegel disk $\Delta = f(\Delta)$ associated to a fixed point z_0.

$\Delta \setminus \{z_0\}$ is foliated by f-invariant circles $(= \varphi(\partial \mathbb{D}_r))$. $, \mathbb{D}_r$

$\overline{P} \cap \Delta \setminus \{z_0\}$ consists of at most finitely many of these circles.

(if there is a $z \in \overline{P} \cap \partial \mathbb{D}_r \Rightarrow \partial \mathbb{D}_r \subset \overline{P}$, while we cannot accumulate from outside, ~~both~~ because $f \sim z \mapsto \lambda z$ on Δ.

Hence if a component U_0 of $\hat{\mathbb{C}} \setminus \overline{P}$, ~~[illegible]~~ intersects $\partial \Delta$, then it must contain a neighborhood of $\partial \Delta$ within Δ

In particular it contains any circle $\partial\Delta_r$ sufficiently close to the boundary.

Similarly, any component V_0 of $f^{-1}(U_0)$ contains every such circle.

We have two cases:

1) $V_0 = U_0 \rightsquigarrow U_0 \subset \mathrm{Fatou}(f)$ (as before)

But $\partial\Delta \subset J(f)$, so $U_0 \cap \partial\Delta = \emptyset$.

Recall: $\partial\Delta \subset J(f)$, comes from the classification of dynamics on hyperbolic Riemann surfaces:

let U_0 be the connected component of $\mathrm{Fatou}(f)$ containing Δ.

then $f(U_0) \subset U_0$, and it cannot be an attracting basin, nor escape to infinity, nor of finite order. Hence it must be an irrational rotation domain, as

2) $V_0 \neq U_0$. As before, $f|_{V_0}$ strictly increases the distance g_{U_0} between nearby points. Hence it maps compact paths to other paths of strictly larger length.

But this is impossible, because V_0 contains $\partial\Delta_r$ close to the boundary, and they are sent diffeomorphically into themselves.

This shows the theorem for fixed points.

In the periodic case, it suffice to apply the argument for a suitable iterate f^m, since $PC(f^m) = PC(f)$

In fact, by chain rule, $C(f^m) = \bigcup_{j=0}^{m-1} f^{-j} C(f)$,

if $z_0 \in C(f)$ and $(z_n)_{n\geq 0}$ is its forward orbit, then $z_n \in PC(f)$ $\forall n \geq 1$.

Write $n = km - j$, $j \in \{0, \dots, m-1\}$. Then $z_n = (f^m)^k(z_{-j})$ $z_{-j} \in f^{-j}(C(f))$, and $z_n \in PC(f^m)$

Similarly if $z \in PC(f^m)$ then $z = f^{mk}(w)$, $w \in f^{-j}(z_0)$, $z_0 \in C(f)$, $j \in \{0, \dots, m-1\}$

Hence $z = z_{mk-j}$, $mk - j > 0 \Rightarrow z \in PC(f)$. □